Exact Solution for a Gyromagnetic Sample and Measurements on a Ferrite*

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Summary—**An outline of an exact solution for a gyromagnetic rod centered in a right circular cylindrical cavity resonator is given. This solution is applied in evaluating dielectric and tensor-magnetic measurements on a well-known ferrite. Complex frequencies and constitutive parameters are introduced and the solution is expanded in series to obtain a convenient calculational scheme. Comparisons are made of exact and perturbation calculations of results from a small and a large sample. The effect of insufficient symmetry of the cavity is discussed and the condition for sufficient symmetry is given. The g value of electrons was 2.02.**

I. Introduction

A METHOD for the measurement of tensor permeability, based on an exact mathematical solution rather than the perturbation solution for a circular cylindrical resonator with a sample, is given. Results obtained using the two-dimensional TM_{nm0} modes of a circular cylindrical resonator are shown. In this case the sample can protrude through holes in the cavity walls, thus permitting very uniform magnetization of the ferrite samples.

An advantage of this exact method is the convenience of measuring the comparatively large frequency and Q changes produced by the samples. Samples were used having diameters of 0.025 and 0.075 inch for measurements at 9200 mc.

It is pertinent to mention that the exact calculations yield directly the constitutive electrical parameters of the sample; *i.e.*, the dielectric constant and the permeability. On the other hand, perturbation calculations yield directly a "demagnetized susceptibility," $(\mu-\mu_0)F(\mu)$, where $F(\mu)$ is a demagnetizing factor and μ is obtained with an additional step. The demagnetized susceptibility and the permeability should not be confused.

II. Solution for Gyromagnetic Sample

An exact solution for a circular cylindrical cavity resonator containing as a sample a centered rod of gyromagnetic material has been given previously.[1] A brief outline is included here, along with several details previously omitted.

The properties of the gyromagnetic material are described first. With the rf magnetic field h and the induction b in right circular cylindrical coordinates, the material is a substance with constitutive parameters which may be expressed as

$$\begin{pmatrix} b_r \\ b_\phi \\ b_z \end{pmatrix} = \begin{pmatrix} \mu & i\alpha & 0 \\ -i\alpha & \mu & 0 \\ 0 & 0 & \mu_z \end{pmatrix} \begin{pmatrix} h_r \\ h_\phi \\ h_z \end{pmatrix}. \tag{1}$$

The dielectric constant ϵ is assumed to be a scalar. The signs of the off-diagonal terms specify that the time dependence is exp $(-i\omega t)$. The antisymmetric matrix (μ) in (1) is identical with the matrix which represents the tensor permeability in the usual Cartesian coordinates. This may be shown by applying rotation operators to the matrix equation $b=(\mu)h$ in Cartesian space and obtaining (1) as a result. The matrix (μ) has an inverse which is written as

$$(\mu)^{-1} = \begin{pmatrix} M & iK & 0 \\ -iK & M & 0 \\ 0 & 0 & M_z \end{pmatrix}. \tag{2}$$

It is introduced because its elements occur most naturally in the wave solutions to be obtained.

To obtain a solution for the cavity with a gyromagnetic rod, the wave equation in the sample must be solved and the waves in it matched to those in the remaining space of the cavity.

The wave equation for a gyromagnetic medium is of the fourth order,[2] for a general three-dimensional wave, or of the second order[3] for two-dimensional waves; *i.e.*, for waves that are independent of one coordinate, here chosen to be the z coordinate. The advantage of working with the two-dimensional waves lies in the simplicity of solution, and they are selected for the remaining work, with the foreknowledge that there exist two-dimensional modes of oscillation for cavity resonators. The two-dimensional wave equation that has solutions which may fit boundary conditions[1] in a closed resonator is

$$(\nabla_p^2 + \omega^2\epsilon/M)E_z = 0, \tag{3}$$

where

$$\nabla_p^2 = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}$$

is the Laplacian in transverse (r, ϕ) space; ω is the angular frequency. Eq. (3) is valid in free space after replacing M by $M_0 \equiv 1/\mu_0$ and ϵ by ϵ_0, where μ_0 and ϵ_0 are the constitutive parameters of free space.

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[1] H. E. Bussey and L. A. Steinert, "Exact solution for a cylindrical cavity containing a gyromagnetic material," PROC. IRE, vol. 45, pp. 693–694; May, 1957.

[2] P. S. Epstein, "Theory of wave propagation in a gyromagnetic medium," *Rev. Mod. Phys.*, vol. 28, pp. 3–17; January, 1956.

[3] M. L. Kales, "Modes in waveguides containing ferrites," *J Appl. Phys.*, vol. 24, pp. 604–608; May, 1953.

The solutions of (3) are of the form

$$E_z = (Ae^{in\phi} + Be^{-in\phi})[FJ_n(\beta r) + GY_n(\beta r)], \quad (4)$$

where $\beta = \omega(\epsilon/M)^{1/2}$, J_n and Y_n are Bessel functions of the first and second kinds, respectively, of inetgral order n, and A, B, F, and G are coefficients to be determined by the boundary conditions. The solutions of (3) in free space are the same as in (4), provided that the cylinder functions have the argument kr where $k = \omega(\epsilon_0/M_0)^{1/2}$. [The time factor exp $(-i\omega t)$ is understood with (4) and other field quantities below.]

Now we consider the boundary value problem of fitting the waves (4) into a right circular cylindrical resonator with a centered gyromagnetic rod (Bussey and Steinert,[1] Fig. 1). The rod has radius a and the cylindrical wall of the cavity is at $r = b$, $b > a$. Since the fields do not vary with z, the height of the assembly is arbitrary.

It is known that in this problem the $e^{+in\phi}$ and $e^{-in\phi}$ solutions have, in general, different resonant frequencies, thus either frequency can be selected for excitation. A choice of signs indicates this selection. The physical or boundary conditions at $r = 0$ and $r = b$ are satisfied with (4) expressed as:

$$E_z = De^{\pm in\phi}J_n(\beta r), \qquad r < a, \quad (5)$$

$$E_z = Ee^{\pm in\phi}C_n(kr), \qquad r > a, \quad (6)$$

where $C_n(kr) = J_n(kr) - Y_n(kr)\,[J_n(kb)/Y_n(kb)]$ is always zero at $r = b$.

The magnetic fields are obtained from (5) and (6) by applying Maxwell's curl equation, $(\mu)^{-1}$ curl $E = i\omega H$. A convenient matrix form for this equation is

$$\begin{pmatrix} M & iK & 0 \\ -iK & M & 0 \\ 0 & 0 & M_z \end{pmatrix} \begin{pmatrix} 0 & -D_z & r^{-1}D_\phi \\ D_z & 0 & -D_r \\ -r^{-1}D_\phi & r^{-1}D_r r & 0 \end{pmatrix} \begin{pmatrix} E_r \\ E_\phi \\ E_z \end{pmatrix} = i\omega \begin{pmatrix} h_r \\ h_\phi \\ h_z \end{pmatrix}, \quad (7)$$

where the D's are derivative operators.

It remains to make E_z and h_ϕ continuous across the boundary at $r = a$. This fixes the ratio of the amplitudes D/E and the value of ω and gives the equation of resonance for the system as:

$$\frac{M\beta aJ_n'(\beta a)}{J_n(\beta a)} \pm nK = \frac{M_0 kaC_n'(ka)}{C_n(ka)}, \quad (8)$$

where the primes indicate derivatives with respect to the arguments. Eq. (8) represents two resonances for $n > 0$. The upper sign $(+nK)$ is used with the Larmor[4] rotating pattern and the lower sign with the anti-Larmor case. When $n = 0$, we have a nondegenerate TM_{0m0} mode and in this mode a small ferrite rod furnishes mostly a dielectric effect on the resonator. In order to evaluate the three unknowns, ϵ, M, and K, three measurements on a ferrite in three different cavity modes are required. It is convenient to make one measurement in a TM_{0m0} resonator and two in a TM_{110} resonator (Larmor and anti-Larmor). A fourth experiment in a TE_{011} cavity can furnish M_z exactly.

[4] Larmor refers to the case where the rf magnetic field pattern rotates in the same sense as the precessing magnetization, and anti-Larmor refers to the opposite sense of rotation.

III. Losses and Complex Arguments

It is apparent that the above equation of resonance (8) does not directly indicate what happens when the frequency of the excitation is not at the resonant frequency. In our experiments, however, excitation away from resonance was carried out to find the Q which indicates the losses; *i.e.*, Q was measured by noting the change in the transmitted power as the frequency departed from resonance. Alternatively, the Q of a resonator may also be measured in the time domain by viewing the exponential decay of the energy, and this, of course, is a resonant oscillation with what appears to be a complex frequency,[5] $\omega^* = \omega' - i\omega''$. In this case, the important feature is that the frequency does not depart from resonance and the equation of resonance is satisfied during the Q measurement.

We will assume that the Q obtained from the variation of transmission with frequency (as it actually was measured) is the same Q obtained from a transient measurement; that the resonant frequency was complex with $\omega^* = \omega' - i\omega'/2Q_F$, where Q_F is the *contributed* Q of the ferrite sample to the resonator, found from $1/Q_F = 1/Q_s - 1/Q_e'$, where Q_s is the resonator Q with the lossy sample, and Q_e' is the Q with the same sample assumed to be lossless. For the present work we will employ the additional widely used assumption that, for a small sample, $Q_e' \doteq Q_e$, the Q of the empty resonator. (Exact expressions for Q_e' of the resonator with a lossless gyromagnetic sample have been developed, but they are rather cumbersome and were not used in the present work.)

IV. Calculation of Results

Eq. (8) may be expressed using relative dielectric and magnetic constants $\epsilon^* = \epsilon/\epsilon_0$, $M^* = \mu_0 M$, and $K^* = \mu_0 K$, where the asterisks also specifically indicate that these quantities are taken to be complex. In addition, we will expand the functions on the left side of (8) in series assuming that a, the sample radius, is small so that to a good approximation, for $n = 1$,

$$M^* \pm K^* = T_\pm(k^*a) + \epsilon^*(k^*a)^2/4 + \epsilon^{*2}(k^*a)^4/96M^*, \quad (9)$$

[5] This may be seen as follows. Consider an electromagnetic field with the time dependence $e^{-i\omega^* t} = e^{-i(\omega' - i\omega'')t} = e^{-\omega'' t}e^{-i\omega' t}$. There is exponential decay for $\omega'' > 0$. Now consider the general definition of $Q = \omega' W/(-dW/dt)$, the stored energy W is divided by the power dissipation per radian. This may be rearranged into the logarithmic differential equation $dW/W = -\omega' dt/Q$. The solution is $W(t) = W(t=0)e^{-\omega' t/Q}$. The field strength, obtained from the square root of the energy, then decays as $e^{-\omega' t/2Q}$. Comparing the two exponential decays, we have a possible physical representation by using the complex $\omega^* = \omega' - i\omega'/2Q$.

where T is the right-hand side of (8) multiplied by μ_0. In (9) we specifically indicate that $k^* = \omega^*(\mu_0\epsilon_0)^{1/2}$ is complex. M^* is first estimated without the last term on the right, then is used in the last term to improve the estimation. A term in $(ka)^6$ can easily be carried in the series.

The benefits of the above expansion especially are apparent during practical calculations, because from (9) we then obtain simple pairs of algebraic equations instead of the difficult four simultaneous transcendental equations that would arise if (8) were solved directly.

Assuming (temporarily) that ϵ^* is known directly from a TM_{0m0} experiment, then it is only necessary to evaluate the TM_{110} data using (9) separated into its real and imaginary parts with $M^* = M' + iM''$, $\epsilon^* = \epsilon' + i\epsilon''$, etc., for exp $(-i\omega t)$ time dependence,

$$M' \pm K' = T_{\pm}' + a^2[\epsilon'(\omega'^2 - \omega''^2) + 2\epsilon''\omega'\omega'']_{\pm}/4c^2 \quad (10)$$

$$M'' \pm K'' = T_{\pm}'' + a^2[\epsilon''(\omega'^2 - \omega''^2) - 2\epsilon'\omega'\omega'']_{\pm}/4c^2, \quad (11)$$

where now we have exhibited only the first term of the expansion; c is the velocity of light. The $\pm$ sign choice indicates, as before, that Larmor and anti-Larmor frequencies enter separately and that each equation represents two equations. T separates into real and imaginary parts as indicated because of the complex frequency in k. The assumption that ϵ^* was known was introduced here for a convenient discussion. In practice, one may iterate the above calculations and use the first estimates of ϵ, M, and K to obtain improved self-consistent results. After calculating the values from (10) and (11), we find μ^* and α^* [the relative constants of the tensor (1)] from

$$\mu^* \pm \alpha^* = (M^* \pm K^*)^{-1}. \quad (12)$$

V. Equipment

A photograph (Fig. 1) shows the microwave source, a Pound stabilized unit, and the magnet. Galvanometers were used as power indicators. An accurate wavemeter was used to measure frequency changes, including Q widths. It has an accurate micrometer movement which seems to repeat its settings to 10^{-5} inches. A table of frequencies for each 10^{-4}-inch change in setting was constructed (using an automatic computer) which made the wave meter convenient to use. Its Q_L (loaded Q) was 16,000 and its transmission was -11 db. Invar was used to give some temperature compensation, although temperature effects are not a major problem when, as here, frequency differences are the main interest.

VI. Dielectric Measurement

Measurements were made of the dielectric and magnetic constants of the ferrite R-1 of General Ceramics. The samples were ground with a diamond wheel from pressed bars (1955 manufacture, dimensions $0.4 \times 0.9 \times 6$ inches) that were cut up.

The TM_{020} resonator for dielectric measurements had an empty frequency f_e of 9860 mc and a Q_L of 8000. The

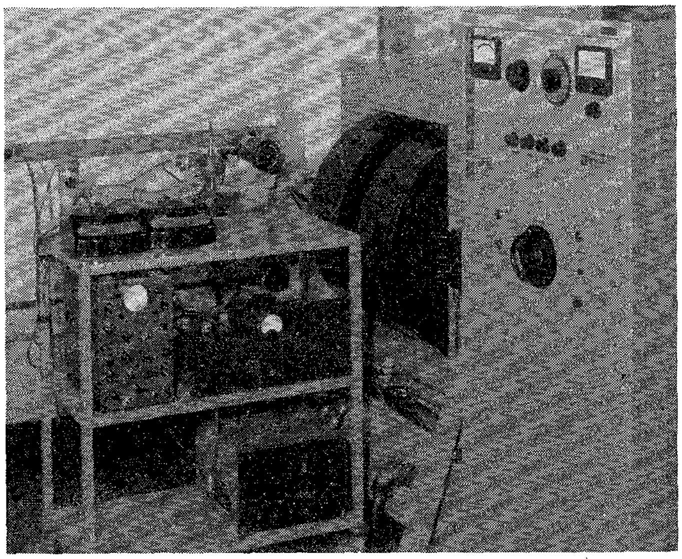

Fig. 1—The experimental assembly.

R-1 sample was 0.075 inch in diameter and 2 inches long. The sample shifted the frequency to 9090 mc which leads to an ϵ' of about 17 by perturbation theory and 11.4 when evaluated by formula (8) with $n=0$ and $\mu' = 0.76$, the unmagnetized value. The value obtained for ϵ'' was 0.013. In this present work, it is assumed that ϵ is independent of the applied dc magnetic field. This, of course, needs to be checked and with the advent of exact calculations a check may be feasible. A convenient expansion of (8) for $n=0$ is $\epsilon^*(k^*a)^2 = 2T/(1-\epsilon^*(k^*a)^2/8M^*)$. For unmagnetized material M^{*-1} is replaced by μ^*.

Since the perturbation method[6] using a rectangular waveguide cavity is widely used for dielectric measurements of ferrites, we will indicate the results from this method. With 0.080-inch holes in each face for inserting the sample, the value of ϵ' is about 12.5. If the holes are eliminated and the sample enclosed, a value near 13.5 is obtained. The holes weaken the electric field, thus compensating some of the error of perturbation theory.[7]

VII. Magnetic Measurement

In this section, tensor permeability measurements on a small (0.025-inch diameter) and a large (0.075-inch diameter) sample rod are compared. In addition, perturbation and exact theory methods of calculating the results are compared. Two different resonators were used for the two sizes of rods. Both resonators had four irises as shown in Fig. 2, but differed in that, for the large sample, excitation of the resonator was through only one iris, whereas the resonator shown in Fig. 2 and used for the small sample was excited through two adjacent irises by waves of equal magnitude (within 2 per

[6] G. Birnbaum and J. Franeau, "Measurements of dielectric constant and loss of solids and liquids by a cavity perturbation method," *J. Appl. Phys.*, vol. 20, pp. 817–818; August, 1949.

[7] An investigation of these matters has been completed and a report will be written. Corrections for the perturbation approximation and hole error were developed.

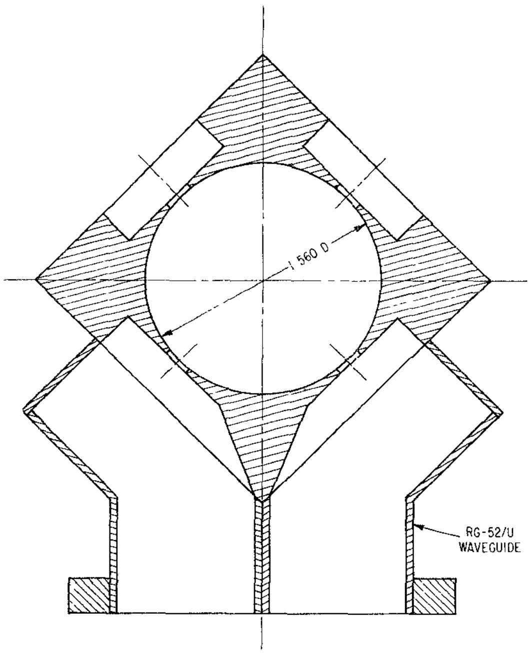

Fig. 2—Cross section of TM_{110} resonator with 4 irises and guides permitting excitation of a rotating field pattern.

cent) and 90° out of phase. This allowed either Larmor or anti-Larmor excitation to be impressed, as was necessary because with a small sample the two modes "overlapped." With the 0.075-inch sample, the two modes were separated sufficiently so that results were obtained with a single input iris. The four irises in this case were used to maintain symmetry.

Both cavities had a length (along the z axis) of 0.9 inch. The small sample was 1.25 inch long and the large sample was 2.0 inch long. This allowed the sample ends to protrude outside the resonator, thus the dc magnetization of the measured part was essentially uniform,[8] except near zero dc field.

The results for the small rod are shown in Fig. 3 and Fig. 4 as evaluated both by the exact solution and by perturbation theory. The results for the large rod are shown in Fig. 5 and Fig. 6. The curves in Fig. 3 and the Larmor curve in Fig. 4 very closely fitted the experimental points, which were omitted for clarity. The anti-Larmor loss data were rather erratic and these points are shown in Fig. 4. [*Note added in proof:* The anti-Larmor loss curve (Fig. 4) is incorrect. The curve probably starts at about 0.0015 unmagnetized and comes down to 0.0002 at 1000 oersteds, remaining at about this value to 8000 oersteds.]

The indications of the rotating-coil Gauss meter were found to be 2.2 per cent too high and, therefore, the abscissa H_{dc} needs to be lowered everywhere by 2.2 per cent.

Comparing the "exactly" calculated small and large rod data, it is seen that the excursions of the curve at resonance are all slightly greater for the large sample. The perturbation approximation introduces relatively little error for the small rod, but considerable error for the large rod. It also artificially displaces the resonance.

From the frequencies of gyromagnetic resonance, 9216 mc and 9114 mc, for the small and large rods, respectively, and with H_{dc} corrected as above, it was found that the g factor of the electrons was 2.013 for the small rod and 2.023 for the large rod, found from $f = geH_{dc}/4\pi$ mc.

The above measurements may be compared with published[9] perturbation results on the same ferrite. The results for the real parts, $\mu' \pm \alpha'$, differ in detail, especially at low values of H_{dc}. The peak absorptions agree to within the inconsistency of the small and large rod data. The g factor obtained from the perturbation rod data[9] would be approximately 1.9; our perturbation results also tend to give g's that are too low.

The missing data in Fig. 3–Fig. 6 were missed due to a resonant absorption and resulting weak signal. It will be observed that the loss factor, $\mu'' + \alpha''$ is not particularly great in the missed region. There are, however, geometrical effects that create a resonant absorption there. These effects may be seen by examining the behavior of the perturbation quantity, $(\mu - \mu_0)F(\mu)$, (see Introduction) where $F(\mu)$ allows for demagnetization, or alternatively, by examining Kittel's resonance formula[10] that allows for demagnetization. It may be shown that these two viewpoints are in agreement.

VIII. Symmetry of the Resonator

Symmetry in the mechanical structure of the resonator is important in tensor permeability measurements. The mathematical solution in Section II assumed that the resonator was perfectly circular. To avoid gross errors, due to initial splitting of the degeneracy by geometrical perturbations, it is sufficient that the symmetry group of the perturbed TM_{110} resonator be C_3 or higher. This means that with irises, loops, or probes around the periphery of the cylinder, there must be three or more identical irises, etc., spaced at equal angles. A circular iris on the axis would be desirable, as it leaves the symmetry C_∞, but is not always practicable.

The deleterious effect of insufficient symmetry was observed with a 9200-mc TM_{110} resonator in which the sine and cosine modes were already split by 5 mc with no sample inserted. Results for the small rod were in general wrong, except that the unmagnetized scalar value was correct and the gyromagnetic resonance was fairly well indicated. However, all of the results for the large rod were very nearly correct.

The resonator with four irises shown in Fig. 2 was annealed and carefully machined and, when empty, the independent sine and cosine modes differed in frequency by only a few hundredths of a megacycle. Furthermore,

[8] R. C. LeCraw and E. G. Spencer, "Tensor permeabilities of ferrites below magnetic saturation," 1956 IRE Convention Record, pt. 5, pp. 66–74.

[9] E. G. Spencer, L. A. Ault, and R. C. LeCraw, "Intrinsic tensor permeabilities on ferrite rods, spheres, and disks," Proc. IRE, vol. 44, pp. 1311–1317; October, 1956.

[10] C. Kittel, "On the theory of ferromagnetic resonance absorption," *Phys. Rev.*, vol. 73, pp. 155–161; January 15, 1948.

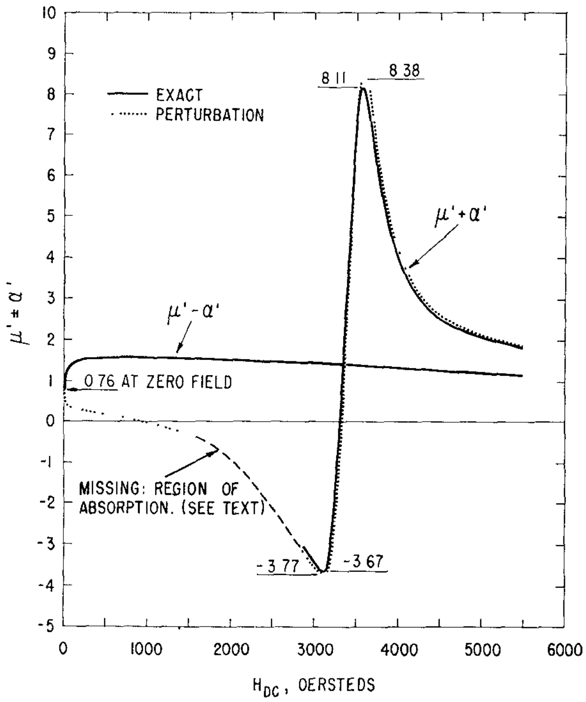

Fig. 3—Circularly polarized relative permeabilities for 0.025-inch-diameter sample.

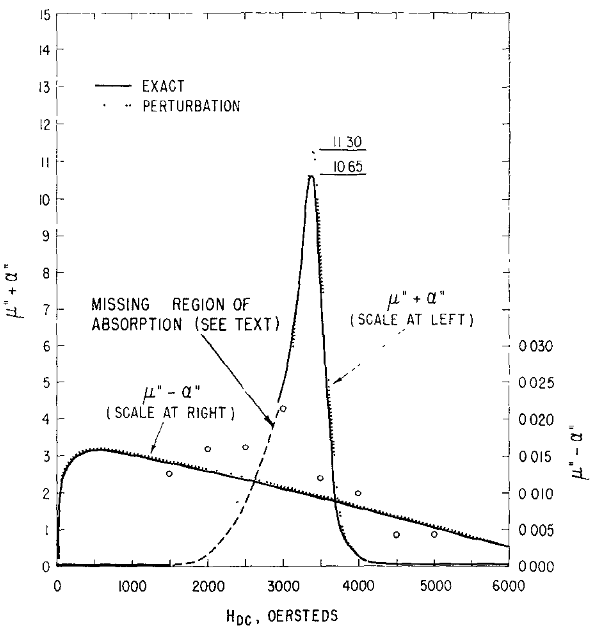

Fig. 4—Circularly polarized relative loss factors for 0.025-inch-diameter sample.

the loaded Q values (about 8000) differed by less than three per cent.

IX. Acknowledgment

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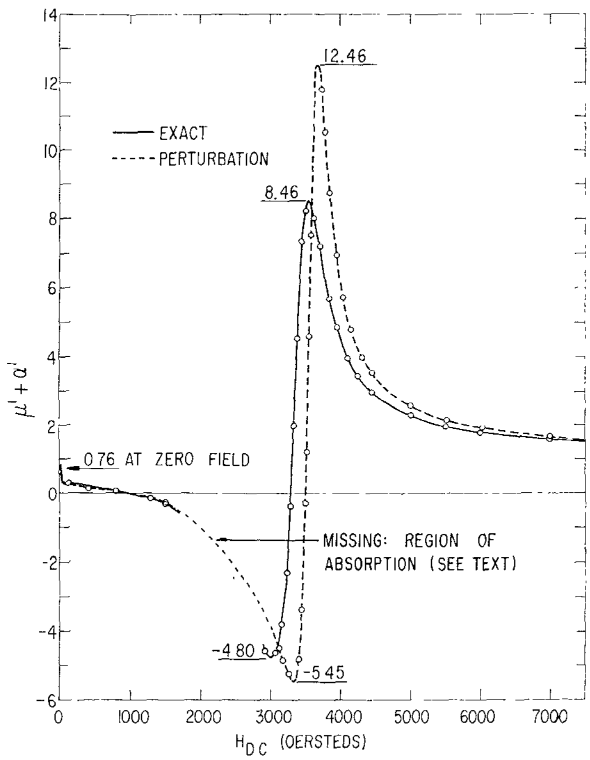

Fig. 5—Circularly polarized relative permeability for 0.075-inch-diameter sample.

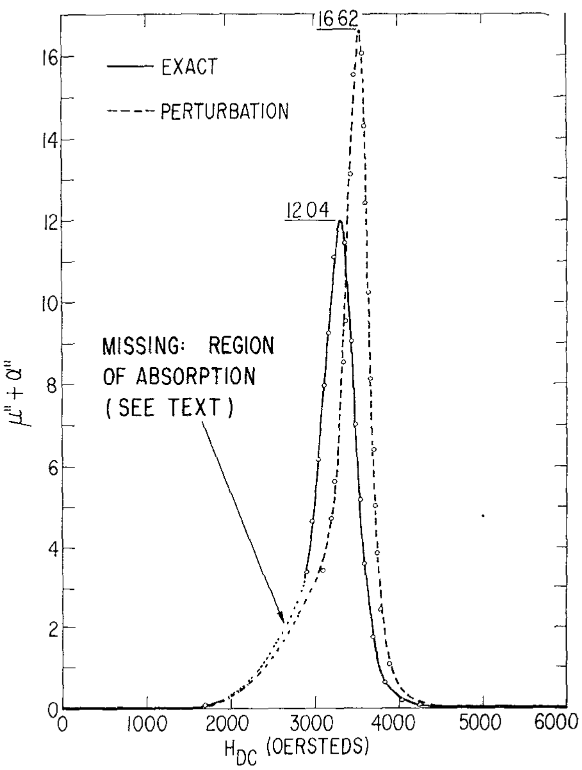

Fig. 6—Circularly polarized relative loss factor for 0.075-inch-diameter sample.